

On some entropic entanglement parameter

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In this paper we present the quantity, which is an entanglement parameter. Its origin is very intriguing, because its construction is motivated by separability criteria based on uncertainty relation. We show that this quantity is asymptotically continuous. We also find the lower and upper bounds for it. Our entanglement parameter has the same feature as the coherent information: both can be negative. There are also some classes of states for which these quantities coincide with each other.

INTRODUCTION

Quantum entanglement is a key feature of composite quantum systems playing a fundamental role in quantum information theory. In contrast to the pure states there are still open problems connected with the classification, characterization and quantification of entanglement of the mixed states (or noisy entanglement). Thus we need many different tools to describe the structure of noisy entanglement. In particular there are various *entanglement measures* which tell us how much entanglement is present in a given state in reference to the singlet - a maximally entangled state. Entanglement measures must fulfill some natural conditions. In particular they should distinguish the nonclassical part of correlation between subsystems from the classical one. Therefore entanglement measures must behave monotonously under LOCC operations. However, there are some functions of states which are not LOCC monotones but give us a hint on the strength of entanglement and reveal some of its characteristic features. We will call them *entanglement parameters*.

Now the question arises, what should we expect from an entanglement parameter? Surely it must be a function, which depends only on a given state. Besides, it must be nonpositive for separable states and so its positivity for a given state indicates that the state is entangled. Entanglement parameters can increase under LOCC, thus they cannot describe entanglement directly, as it would imply that entanglement can be increased by classical means. However, it is plausible that normalized entanglement parameters simply underestimate some normalized entanglement measures, where normalized entanglement parameter (measure) means it is equal to 1 for the singlet state. Let us consider in this context an entanglement parameter which is called the *coherent information* [1, 2] and its maximum value attainable by LOCC. The maximal value is already an entanglement measure. (Notice, that from every entanglement parameter one can obtain entanglement measure as the maximal value attainable by LOCC.) One can show, that the maximal value of the coherent information does not exceed $\log_2 d$, i.e. the value on the singlet state. Thus the coherent information can only underestimate the value of the above entanglement measure. Let us note that it is important to know the maximal value of the entanglement measure induced by a given parameter on $\mathcal{C}^2 \otimes \mathcal{C}^2$, so that we have a reference point.

Let us recall some known entanglement parameters (i.e. functions which fulfill the above requirements). A useful entanglement parameter $M(\rho)$ was introduced which characterizes the maximal violation of the Bell-CHSH inequality for arbitrary mixed two-qubit states [3]. It depends only on some state parameters and contains all the information that is needed to decide whether a state violates the Bell-CHSH inequality. A closely related entanglement parameter was defined [4] as a measure of violation of the Bell-CHSH inequality $B(\rho) = \sqrt{\max\{0, M(\rho) - 1\}}$, which for an arbitrary two-qubit pure state equals to measures of entanglement *negativity* [5, 6] and *concurrence* [7, 8]. There is another entanglement parameter $N(\rho)$ defined for an arbitrary two-qubit state connected with possibility of teleportation by use of a given state ρ as a quantum channel [9]. In particular it has been shown that every two-qubit state, which violates the Bell-CHSH inequality offers better fidelity of teleportation than the pure classical channel. Perhaps the most useful entanglement parameter in quantum communication is the coherent information, which is closely related to the conditional entropy.

In this paper we introduce a new entanglement parameter. Its construction is inspired by the separability criteria considered in [10, 11], which are based on uncertainty relation, where the concept of detection of entanglement is as follows. One or several observables M_i are taken and the sum of entropies $\sum_i S(M_i)_\rho$ or the sum of variances $\sum_i \delta(M_i)_\rho$. For product states a lower bound is derived for this sum, which by concavity also holds for separable states. Thus violation of this lower bound for a given state ρ implies that ρ is entangled. Notice that these entropic separability criteria, are different from the other approaches (see for example [2, 12, 13, 14, 15, 16]), because here only the probability distribution of the outcomes of a measurement is taken into account, and not the eigenvalues of the density matrix.

Our parameter is based on the simplest possible separability criterion emerging from the above concept, and is given by the following formula:

$$\mathcal{M}(\varrho) = \sup_{\mathcal{P}} \left(\inf_{\delta \in \mathcal{PS}} H(\delta, \mathcal{P}) - H(\varrho, \mathcal{P}) \right) \quad (1)$$

where \mathcal{PS} is the set of product states, $\mathcal{P} \equiv \{P_i\}$ is a set of projectors representing a von Neumann measurement and $H(\varrho, \mathcal{P})$ is the Klein entropy, which is equal to the Shannon entropy for the probability distribution $p_i = \text{Tr} \varrho P_i$ and $H(\{p_i\}) = -\sum_i p_i \log_2 p_i$. One could here consider also Renyi entropies, however in this paper we concentrate only on the Shannon one, basing on experience [2, 17] that quantities built out of the Shannon and von Neumann entropies often have operational interpretation.

The quantity $\mathcal{M}(\varrho)$, which we will call the *entropic entanglement parameter* (\mathcal{E} -parameter), has interesting features. For all separable states $\mathcal{M}(\varrho)$ is nonpositive. It tells us about the "strength" of entanglement, because if we want to obtain positive value of $\mathcal{M}(\varrho)$ we need a state having enough amount of entanglement. For instance for part of the entangled isotropic states \mathcal{E} -parameter is nonpositive. However, the \mathcal{E} -parameter also "feels" some other feature of entangled states, because it relates the greatest difference between entropies of a "nearest" separable or product state and a given state after making measurement, so it must somehow see the structure of the state and be connected with complementarity between eigenbasis of an entangled state and eigenbasis of a product state.

\mathcal{E} -PARAMETER FOR SEPARABLE AND PRODUCT STATES

In this section we show that our \mathcal{E} -parameter distinguishes the separable states, because for them the \mathcal{E} -parameter is always negative or equal to zero.

Proposition 1 *For every separable state ϱ_{sep} the \mathcal{E} -parameter $\mathcal{M}(\varrho_{sep})$ is less or equal to zero.*

$$\mathcal{M}(\varrho_{sep}) \leq 0 \quad (2)$$

Proof.

Let \mathcal{P} be a set of projectors representing a measurement. We know that every separable state can be written as a convex mixture of product states and the Klein entropy for a given measurement is a concave function. These facts imply that for every separable state ϱ_{sep} we can find a product state ϱ_{prod} such that

$$H(\varrho_{prod}, \mathcal{P}) \leq H(\varrho_{sep}, \mathcal{P}) \quad (3)$$

So

$$\inf_{\delta \in \mathcal{PS}} H(\delta, \mathcal{P}) = \inf_{\delta \in \mathcal{SE}\mathcal{P}} H(\delta, \mathcal{P}) \quad (4)$$

where $\mathcal{SE}\mathcal{P}$ is the set of separable states. Then for a given measurement \mathcal{P}

$$\inf_{\delta \in \mathcal{PS}} H(\delta, \mathcal{P}) - H(\varrho_{sep}, \mathcal{P}) = \inf_{\delta \in \mathcal{SE}\mathcal{P}} H(\delta, \mathcal{P}) - H(\varrho_{sep}, \mathcal{P}) \leq 0 \quad (5)$$

It turns out that in particular the \mathcal{E} -parameter is equal to zero for all pure product states, which we are going to prove below.

Proposition 2 *For every product pure state ϱ_{prod} the \mathcal{E} -parameter $\mathcal{M}(\varrho_{prod})$ is equal to zero.*

$$\mathcal{M}(\varrho_{prod}) = 0 \quad (6)$$

Proof.

From the lemma above we know that

$$\mathcal{M}(\varrho_{prod}) \leq 0 \quad (7)$$

Lets consider a measurement $\tilde{\mathcal{P}}$ which is in eigenbasis of state ϱ_{prod} . Then

$$\mathcal{M}(\varrho_{prod}) \geq \inf_{\delta \in \mathcal{PS}} H(\delta, \tilde{\mathcal{P}}) - H(\varrho_{prod}, \tilde{\mathcal{P}}) = \inf_{\delta \in \mathcal{PS}} H(\delta, \tilde{\mathcal{P}}) = 0 \quad (8)$$

The inequalities (7) and (8) imply that

$$\mathcal{M}(\varrho_{prod}) = 0 \quad (9)$$

UPPER BOUND FOR \mathcal{M}

In this section we present the upper bounds for our quantity. Some of them are general and some are applicable for a fixed dimension.

Proposition 3 *Let ϱ be a state acting on Hilbert space $\mathcal{C}^d \otimes \mathcal{C}^d$. Then $\mathcal{M}(\varrho)$ is bounded from above by information content $I(\varrho)$ of state ϱ .*

$$\mathcal{M}(\varrho) \leq I(\varrho) = 2 \log_2 d - S(\varrho) \quad (10)$$

Proof.

We use the fact that for any state ϱ we have

$$H(\varrho, \mathcal{P}) \geq S(\varrho) \quad \text{and} \quad H(\varrho, \mathcal{P}) \leq 2 \log_2 d. \quad (11)$$

Then

$$\mathcal{M}(\varrho) = \sup_{\mathcal{P}} (\inf_{\delta \in \mathcal{PS}} H(\delta, \mathcal{P}) - H(\varrho, \mathcal{P})) \leq \sup_{\mathcal{P}} \inf_{\delta \in \mathcal{PS}} H(\delta, \mathcal{P}) - S(\varrho) \leq 2 \log_2 d - S(\varrho) \quad (12)$$

Remark

Notice that this bound is not very good, because for all states it is greater or equal to zero. So it does not "see" the difference between a separable state and an entangled one. Thus this bound can not give us any information about the structure of a state. However, it is natural and has an operational meaning, because it is equal to information content of state. Below we present a little better bound for \mathcal{E} -parameter.

Proposition 4 *Let ϱ be a state acting on Hilbert space $\mathcal{C}^d \otimes \mathcal{C}^d$. Then $\mathcal{M}(\varrho)$ is bounded from above as follows:*

$$\mathcal{M}(\varrho) \leq \log_2 d + (1 - \frac{1}{d}) \log_2(d+1) - S(\varrho) \quad (13)$$

Proof.

We will try to estimate the quantity

$$\sup_{\mathcal{P}} \inf_{\delta \in \mathcal{PS}} H(\delta, \mathcal{P}) \quad (14)$$

Let $\mathcal{P} = \{P_i\}$ and $\sigma_{\mathcal{P}}$ be a product state having the greatest projection $p = \text{Tr} P_k \sigma_{\mathcal{P}}$ on the the least entangled projectors P_k from $\{P_i\}$. Then

$$\sup_{\mathcal{P}} \inf_{\delta \in \mathcal{PS}} H(\delta, \mathcal{P}) \leq \sup_{\mathcal{P}} H(\sigma_{\mathcal{P}}, \mathcal{P}) \quad (15)$$

Notice that p is equal to the square of the greatest Schmidt coefficient of state φ_k , where $P_k = |\varphi_k\rangle\langle\varphi_k|$, so $\frac{1}{d} \leq p \leq 1$. Then

$$\sup_{\mathcal{P}} H(\sigma_{\mathcal{P}}, \mathcal{P}) = \sup_{\frac{1}{d} \leq p \leq 1} H(p, \frac{1-p}{d^2-1}, \frac{1-p}{d^2-1}, \dots, \frac{1-p}{d^2-1},) = \quad (16)$$

$$H(\frac{1}{d}, \frac{1}{d^2+d}, \frac{1}{d^2+d}, \dots, \frac{1}{d^2+d}) = \log_2 d + (1 - \frac{1}{d}) \log_2(d+1) \quad (17)$$

Using the same arguments as in the proof of the previous proposition we get

$$\mathcal{M}(\varrho) \leq \sup_{\mathcal{P}} (\inf_{\sigma_{\mathcal{P}}} H(\sigma_{\mathcal{P}}, \mathcal{P}) - H(\varrho, \mathcal{P})) \leq \sup_{\mathcal{P}} \inf_{\sigma_{\mathcal{P}}} H(\sigma_{\mathcal{P}}, \mathcal{P}) - S(\varrho) \leq \log_2 d + (1 - \frac{1}{d}) \log_2(d+1) - S(\varrho) \quad (18)$$

Remark. Notice that this bound is nonpositive for the maximally mixed state.

$$\mathcal{M}(\frac{I}{d^2}) \leq -\log_2 d + (1 - \frac{1}{d}) \log_2(d+1) < 0 \quad (19)$$

Proposition 5 Let ϱ be a state acting on Hilbert space $\mathcal{C}^2 \otimes \mathcal{C}^2$. Then

$$\mathcal{M}(\varrho) \leq 1 - S(\varrho) \quad (20)$$

First we show the following fact:

Fact 1 In any 2-dimensional subspace of space $\mathcal{C}^2 \otimes \mathcal{C}^2$ we can find a product state.

Proof. Let $|\psi_1\rangle$ and $|\psi_2\rangle$ be states spanning 2-dimensional subspaces.

$$\psi_1 = \sum_{ij}^2 a_{ij} |ij\rangle \quad \psi_2 = \sum_{ij}^2 b_{ij} |ij\rangle \quad (21)$$

Let $A = \{a_{ij}\}$ and $B = \{b_{ij}\}$. We show that we can find a product state in $\text{span}(|\psi_1\rangle, |\psi_2\rangle)$ i.e

$$\exists_{\alpha, \beta} \quad \alpha |\psi_1\rangle + \beta |\psi_2\rangle = \varphi_{\text{prod}} \quad (22)$$

In the space $\mathcal{C}^2 \otimes \mathcal{C}^2$ equation (22) is equivalent to the following conditions:

$$\exists_{\alpha, \beta} \quad r[\alpha A + \beta B] = 1 \iff \exists_{\alpha, \beta} \quad \text{Det}[\alpha A + \beta B] = 0 \quad (23)$$

Notice that

$$\text{Det}[\alpha A + \beta B] = 0 \iff \alpha \text{Det}[A + \gamma B] = 0 \iff \text{Det}[A + \gamma B] = 0 \quad (24)$$

where $\gamma = \frac{\beta}{\alpha}$. After elementary calculations we get the following equation:

$$\text{Det}[B]\gamma^2 + c\gamma + d = 0 \quad (25)$$

Notice that if B is not a matrix representing a product state then $\text{Det}[B] \neq 0$. So equation (25) always has the solution, because it is the square equation. In the opposite case, if $\text{Det}[B] = 0$ and $|\psi_2\rangle$ is a product state, we immediately have a product state in $\text{span}(|\psi_1\rangle, |\psi_2\rangle)$.

Proof (of Proposition 5). Fact 1 implies that for any measurement $\mathcal{P} \in \mathcal{C}^2 \otimes \mathcal{C}^2$, $\mathcal{P} = \{P_i\}_{i=1}^4$ there exists a product state $\delta_{\text{prod}}^{\mathcal{P}} \in \text{span}\{P_k, P_l\}$ ($k, l \in \{1, 2, 3, 4\}$, $k \neq l$). So

$$\inf_{\delta \in \mathcal{PS}} H(\delta, \mathcal{P}) \leq H(\delta_{\text{prod}}^{\mathcal{P}}, \mathcal{P}) \leq 1 \quad (26)$$

and

$$\sup_{\mathcal{P}} (\inf_{\delta \in \mathcal{PS}} H(\delta, \mathcal{P})) = 1 \quad (27)$$

where supremum is achievable in the Bell basis. Then

$$\mathcal{M}(\varrho) = \sup_{\mathcal{P}} (\inf_{\delta \in \mathcal{PS}} H(\delta, \mathcal{P}) - H(\varrho, \mathcal{P})) \leq \sup_{\mathcal{P}} \inf_{\delta \in \mathcal{PS}} H(\delta, \mathcal{P}) - S(\varrho) = 1 - S(\varrho) \quad (28)$$

Remark. For $\mathcal{C}^3 \otimes \mathcal{C}^3$ we have numerical result saying that

$$\sup_{\mathcal{P}} (\inf_{\delta \in \mathcal{PS}} H(\delta, \mathcal{P})) \approx 1.71 \quad (29)$$

It implies (analogously to above proposition) that

$$\mathcal{M}(\varrho) \leq 1.71 - S(\varrho) \quad (30)$$

LOWER BOUNDS FOR \mathcal{E} -PARAMETER

Let us now pass to the lower bounds, which we obtained for the value of \mathcal{E} -parameter.

Proposition 6 *Let ϱ^B be a state diagonal in a maximally entangled basis on Hilbert space $\mathcal{C}^d \otimes \mathcal{C}^d$. Then*

$$\mathcal{M}(\varrho^B) \geq \log_2 d - S(\varrho^B) = I_{coh}(\varrho^B) \quad (31)$$

where I_{coh} is the coherent information equal to a difference between the von Neumann entropy of subsystem and entropy of entire state.

Proof.

We know that for any state ϱ if measurement \mathcal{P} is made in the eigenbasis of the state ϱ then the Klein entropy $H(\varrho, \mathcal{P})$ is equal to the von Neumann entropy of the state. A basis consisting of maximally entangled projectors $\mathcal{P}^B \equiv \{P_i^B\}$ (so called Bell basis) is eigenbasis of state ϱ^B . So

$$\mathcal{M}(\varrho^B) = \sup_{\mathcal{P}} (\inf_{\delta \in \mathcal{P}\mathcal{S}} H(\delta, \mathcal{P}) - H(\varrho^B, \mathcal{P})) \geq \inf_{\delta \in \mathcal{P}\mathcal{S}} H(\delta, \mathcal{P}^B) - S(\varrho^B) \quad (32)$$

Now, we have to show that $\inf_{\delta \in \mathcal{P}\mathcal{S}} H(\delta, \mathcal{P}^B) = \log_2 d$. First notice that $p_i = \text{Tr} P_i^B \sigma \in [0, \frac{1}{d}]$ for any product state σ . Notice that above restriction implies that at least d of probabilities p_i must be nonzero and if exactly d are nonzero then we have $\{p_i\} = \{\frac{1}{d}, \frac{1}{d}, \dots, \frac{1}{d}\}$. Here we can recall the following fact [18], saying that any change toward equalization of probabilities p_1, p_2, \dots, p_n increases H and if $p_1 < p_2$ and we increase p_1 , decreasing p_2 by an equal amount so that p_1 and p_2 are more nearly equal, then H increases. This fact implies that $H(\{\frac{1}{d}, \frac{1}{d}, \dots, \frac{1}{d} - \delta, \delta\}) > H(\{\frac{1}{d}, \frac{1}{d}, \dots, \frac{1}{d}\}) = \log_2 d$, where $\delta \in (0, \frac{1}{d})$. By the induction we get that the smallest entropy attainable under given restriction is $\log_2 d$. So we have

$$\mathcal{M}(\varrho^B) \geq \log_2 d - S(\varrho^B) \quad (33)$$

Remark 1. In general, we have a lower bound which is true for any state:

$$\mathcal{M}(\varrho) \geq \inf_{\delta \in \mathcal{P}\mathcal{S}} H(\delta, \mathcal{P}^e) - S(\varrho) \quad (34)$$

where \mathcal{P}^e represents the measurement made in eigenbasis of state ϱ .

Lemma 1 *Let c be the greatest Schmidt coefficient of the state φ acting on the Hilbert space $\mathcal{C}^d \otimes \mathcal{C}^d$. If we are able to construct the eigenbasis $\{|\psi_i\rangle\}$ of $|\psi\rangle$, i.e. $|\psi\rangle \in \{|\psi_i\rangle\}$ such that the greatest Schmidt coefficient of each ψ_i is less or equal to c , then the following inequality holds*

$$\inf_{\sigma_{prod}} H(\sigma_{prod}, P) \geq -kc \log_2 c - (1 - kc) \log_2(1 - kc) = H(c, \dots, c, 1 - kc) \quad (35)$$

where infimum is taken over all product states σ_{prod} , k is equal to $\lfloor \frac{1}{c} \rfloor$ and k is the number of c .

Proof.

Notice that the maximal overlap between the state ψ_i and any product state σ_{prod} is equal to the square of the greatest Schmidt coefficient of ψ_i [19]. Let $p_i = \text{Tr} P_i \sigma$, then every probability p_i is bounded by c i.e. $p_i \leq c$. If we want to have the smallest entropy $H(\{p_i\})$ we must have as many of the probabilities p_i equal to 0 or c as possible. There may be at most $\lfloor \frac{1}{c} \rfloor$ probabilities p_i equal to c . These conditions are connected with the concavity of entropy and the lemma that any change of probabilities towards equalization increases entropy, which implies that

- 1) $H(c, \dots, c, 1 - kc) \leq H(c - \delta, \dots, c, 1 - kc, \delta)$
- 2) $H(c, \dots, c, 1 - kc) \leq H(c - \delta, \dots, c, 1 - kc + \delta)$

By using the condition 1) and 2) and the induction rule we show that entropy $H(c, \dots, c, 1 - kc)$ is the least achievable for probability distribution with all $p_i \leq c$.

Proposition 7 *Let $|\varphi\rangle$ be a pure state acting on the Hilbert space $\mathcal{C}^d \otimes \mathcal{C}^d$ where $d=2,4,8$. Then*

$$\mathcal{M}(\varphi) \geq H(c, \dots, c, 1 - kc) \quad (36)$$

where $c = \max\{a_i^2\}$ for $|\varphi\rangle = \sum_i a_i |e_i\rangle \otimes |f_j\rangle$, k is the number of probabilities equal to c and $k = \lfloor \frac{1}{c} \rfloor$.

Proof

For the state $|\varphi\rangle = \sum_i a_i |e_i\rangle \otimes |f_j\rangle$ acting on the Hilbert space $\mathcal{C}^d \otimes \mathcal{C}^d$ (where $d = 2, 4, 8$), we are able to construct a measurement $P^\varphi = \{|\varphi_k\rangle\langle\varphi_k|\}$ representing the eigenbasis of the state $|\varphi\rangle$ such that every Schmidt coefficient of the vector $|\varphi_i\rangle$ is less or equal to the greatest coefficient of $|\varphi\rangle$. Then by lemma 1 we have

$$\mathcal{M}(\varphi) = \sup_{\mathcal{P}} \left(\inf_{\delta \in \mathcal{PS}} H(\delta, \mathcal{P}) - H(\varphi, \mathcal{P}) \right) \geq \inf_{\delta \in \mathcal{PS}} H(\delta, P^\varphi) \geq H(c, \dots, c, 1 - kc) \quad (37)$$

Our basis will consist of the vector $|\varphi\rangle$ and other vectors $|\varphi_k\rangle$ with the same (with regard to absolute value) set of Schmidt coefficients in the basis $\{|e_i\rangle \otimes |f_j\rangle\}$, where we choose such representation of the state $|\varphi\rangle$ that $a_i \in \mathcal{R}$. Then we can construct an eigenbasis of $|\varphi\rangle$ consisting of the vectors $|\varphi\rangle$ and other vector $|\varphi_k\rangle$ with the same (with regard to the absolute value) set of Schmidt coefficients in the basis $\{|e_i\rangle \otimes |f_j\rangle\}$. In the case $\mathcal{C}^2 \otimes \mathcal{C}^2$ we can express the basis $\{|\varphi_k\rangle\}$ as the following matrix:

$$\begin{pmatrix} a_1 & 0 & 0 & a_2 \\ -a_2 & 0 & 0 & a_1 \\ 0 & a_1 & a_2 & 0 \\ 0 & -a_2 & a_1 & 0 \end{pmatrix}$$

The analogous matrix representing the eigenbasis of $|\varphi\rangle$ for the Hilbert space $\mathcal{C}^4 \otimes \mathcal{C}^4$ is of the form:

$$\begin{pmatrix} a_1 & 0 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 & 0 & a_3 & 0 & 0 & 0 & 0 & a_4 \\ -a_2 & 0 & 0 & 0 & 0 & a_1 & 0 & 0 & 0 & 0 & a_4 & 0 & 0 & 0 & 0 & -a_3 \\ a_3 & 0 & 0 & 0 & 0 & a_4 & 0 & 0 & 0 & 0 & -a_1 & 0 & 0 & 0 & 0 & -a_2 \\ a_4 & 0 & 0 & 0 & 0 & -a_3 & 0 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 & 0 & -a_1 \\ 0 & a_1 & 0 & 0 & a_2 & 0 & 0 & 0 & 0 & 0 & 0 & a_3 & 0 & 0 & 0 & a_4 \\ 0 & -a_2 & 0 & 0 & a_1 & 0 & 0 & 0 & 0 & 0 & 0 & a_4 & 0 & 0 & 0 & -a_3 \\ 0 & a_3 & 0 & 0 & a_4 & 0 & 0 & 0 & 0 & 0 & 0 & -a_1 & 0 & 0 & 0 & -a_2 \\ 0 & a_4 & 0 & 0 & -a_3 & 0 & 0 & 0 & 0 & 0 & 0 & a_2 & 0 & 0 & 0 & -a_1 \\ 0 & 0 & a_1 & 0 & 0 & 0 & 0 & a_2 & a_3 & 0 & 0 & 0 & 0 & a_4 & 0 & 0 \\ 0 & 0 & -a_2 & 0 & 0 & 0 & 0 & a_1 & a_4 & 0 & 0 & 0 & 0 & -a_3 & 0 & 0 \\ 0 & 0 & a_3 & 0 & 0 & 0 & 0 & a_4 & -a_1 & 0 & 0 & 0 & 0 & -a_2 & 0 & 0 \\ 0 & 0 & a_4 & 0 & 0 & 0 & 0 & -a_3 & a_2 & 0 & 0 & 0 & 0 & -a_1 & 0 & 0 \\ 0 & 0 & 0 & a_1 & 0 & 0 & a_2 & 0 & 0 & a_3 & 0 & 0 & a_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & -a_2 & 0 & 0 & a_1 & 0 & 0 & a_4 & 0 & 0 & -a_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_3 & 0 & 0 & a_4 & 0 & 0 & -a_1 & 0 & 0 & -a_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_4 & 0 & 0 & -a_3 & 0 & 0 & a_2 & 0 & 0 & -a_1 & 0 & 0 & 0 \end{pmatrix}$$

For $\mathcal{C}^8 \otimes \mathcal{C}^8$ this basis will consist of 8 groups of vectors each spanning an orthogonal subspace. Values of non-zero coefficients of the vectors of every group rewritten in basis $\{|e_i\rangle \otimes |f_j\rangle\}$ represents the following matrix:

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \\ a_2 & -a_1 & -a_4 & a_3 & -a_6 & a_5 & -a_8 & a_7 \\ a_3 & a_4 & -a_1 & -a_2 & a_7 & -a_8 & -a_5 & a_6 \\ -a_4 & a_3 & -a_2 & a_1 & a_8 & a_7 & -a_6 & -a_5 \\ -a_5 & -a_6 & a_7 & -a_8 & a_1 & a_2 & -a_3 & a_4 \\ -a_6 & a_5 & -a_8 & -a_7 & -a_2 & a_1 & a_4 & a_3 \\ a_7 & a_8 & a_5 & -a_6 & -a_3 & a_4 & -a_1 & -a_2 \\ -a_8 & a_7 & a_6 & a_5 & -a_4 & -a_3 & -a_2 & a_1 \end{pmatrix}$$

Remark 1. In particular as a consequence of proposition 7 for any pure state $|\varphi\rangle$ acting on the Hilbert space $\mathcal{C}^2 \otimes \mathcal{C}^2$ we get the following lower bound for \mathcal{E} -parameter.

$$\mathcal{M}(\varphi) \geq E(\varphi) = S_R(\varphi) \quad (38)$$

where E is entanglement measure for pure bipartite state, which is equal to $S_R(\varphi)$ - the von Neumann entropy of reduced density matrix $\text{Tr}_B |\varphi\rangle\langle\varphi| = \text{Tr}_A |\varphi\rangle\langle\varphi|$.

Remark 2. Notice that this lower bound is not greater than entropy of subsystem of state $|\varphi\rangle$, so is not greater than coherent information.

RESULTS FOR SOME FAMILIES OF STATES

Evaluating \mathcal{E} -parameter is very difficult, because its definition is a kind of so called "minmax". But there are some classes of states, in particular states with high symmetry, for which we are able to find the exact value of \mathcal{M} .

For states diagonal in maximally entangled basis consisting of the vectors acting on Hilbert space $\mathcal{C}^2 \otimes \mathcal{C}^2$, we know the value of \mathcal{E} -parameter:

$$\mathcal{M}(\varrho^B) = 1 - S(\varrho^B) \quad (39)$$

which in particular gives a result for maximally entangled state ψ_+^2

$$\mathcal{M}(\psi_+^2) = 1 \quad (40)$$

This result follows from combining the lemma 6 with the lemma 5.

For maximally entangled states in higher dimension than $d=2$ we only have a lower bound

$$\mathcal{M}(\psi_+^d) \geq \log_2 d \quad (41)$$

For $d = 3$ we have numerical result

$$\mathcal{M}(\psi_+^3) \approx 1.663 > \log_2 3 \quad (42)$$

We know the value of \mathcal{E} -parameter for $\mathcal{C}^2 \otimes \mathcal{C}^2$ isotropic states ϱ_{iso} , which belong to a subset of Bell diagonal states:

$$\mathcal{M}(\varrho_{iso}) = 1 - S(\varrho^{iso}) \quad (43)$$

where the isotropic states are of the form

$$\varrho_{iso} = pP_+ + \frac{1-p}{d^2}I \quad \lambda \in [0, 1] \quad (44)$$

and P_+ is maximally entangled state and I is the identity matrix.

Figure 1 shows the value of \mathcal{E} -parameter for the isotropic states. We can see that for the separable isotropic states $\mathcal{M} < 0$, but what interesting for a part of the entangled isotropic states \mathcal{M} is negative. So it means that a given state must have "enough" entanglement to have positive value of our entanglement parameter.

Negative result. We have suspected that for a pure state φ , we get $\mathcal{M}(\varphi) = S_A(\varphi)$. But numerical calculations show that there exist such states (and for a set of randomly chosen states it turns out to be the majority of them) for which

$$\mathcal{M}(\varphi) \neq S_A(\varphi) \quad (45)$$

ASYMPTOTIC CONTINUITY OF \mathcal{M}

Our entanglement parameter has a feature, which is especially useful in the regime of many copies, i.e. asymptotic continuity.

Theorem 1 *For any state ϱ the quantity $\mathcal{M}(\varrho)$ is asymptotically continuous, which refers the condition*

$$\forall_{\varrho_1, \varrho_2} |\mathcal{M}(\varrho_1) - \mathcal{M}(\varrho_2)| \leq K\varepsilon \log_2 d + C \quad (46)$$

where C is constant and $\varepsilon = \|\varrho_1 - \varrho_2\|$.

Proof. We show that \mathcal{M} is "robust under admixture" i.e

$$|(\mathcal{M}(\varrho) - \mathcal{M}((1-\varepsilon)\varrho + \varepsilon\sigma))| \leq 4\varepsilon \log_2 d + H(\varepsilon) \quad (47)$$

This feature is equivalent to asymptotic continuity, which is proven in paper [20].

Before we pass to the proof of the theorem we need to introduce the following lemma:

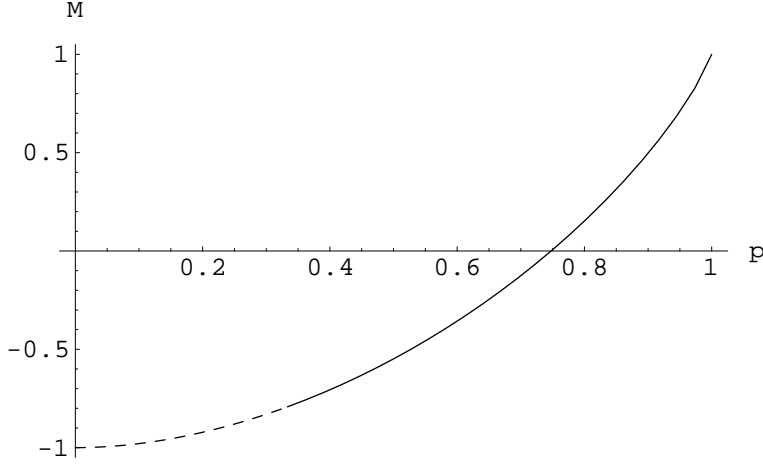


FIG. 1: The dashed line represents the value of \mathcal{E} -parameter for the separable isotropic states and the solid one the value for entangled isotropic states.

Lemma 2 *Let \mathcal{P} be a given measurement. Then*

$$|\mathcal{M}(\varrho, \mathcal{P}) - \mathcal{M}((1 - \varepsilon)\varrho + \varepsilon\sigma, \mathcal{P})| \leq 4\varepsilon \log_2 d + H(\varepsilon) \quad (48)$$

where

$$\mathcal{M}(\varrho, \mathcal{P}) = \inf_{\delta \in \mathcal{P}_S} H(\delta, \mathcal{P}) - H(\varrho, \mathcal{P}) \quad (49)$$

Proof.

$$\begin{aligned} |\mathcal{M}(\varrho, \mathcal{P}) - \mathcal{M}((1 - \varepsilon)\varrho + \varepsilon\sigma, \mathcal{P})| &= \left| \inf_{\delta \in \mathcal{P}_S} H(\delta, \mathcal{P}) - H(\varrho, \mathcal{P}) - \inf_{\delta \in \mathcal{P}_S} H(\delta, \mathcal{P}) + H((1 - \varepsilon)\varrho + \varepsilon\sigma, \mathcal{P}) \right| \\ &= |H((1 - \varepsilon)\varrho + \varepsilon\sigma, \mathcal{P}) - H(\varrho, \mathcal{P})| = |H((1 - \varepsilon)\varrho + \varepsilon\sigma, \mathcal{P}) - (1 - \varepsilon)H(\varrho, \mathcal{P}) - \varepsilon H(\sigma, \mathcal{P}) + \varepsilon H(\varrho, \mathcal{P}) + \varepsilon H(\sigma, \mathcal{P})| \\ &\leq |H((1 - \varepsilon)\varrho + \varepsilon\sigma, \mathcal{P}) - (1 - \varepsilon)H(\varrho, \mathcal{P}) - \varepsilon H(\sigma, \mathcal{P})| + \varepsilon |H(\varrho, \mathcal{P})| + \varepsilon |H(\sigma, \mathcal{P})| \leq H(\varepsilon) + 4\varepsilon \log_2 d \end{aligned} \quad (50)$$

We use here the facts that $|H(\varrho)| \leq 2 \log_2 d$ and

$$\sum_k p_k H(\varrho_k, \mathcal{P}) \leq H(\sum_k p_k \varrho_k, \mathcal{P}) \leq \sum_k p_k H(\varrho_k, \mathcal{P}) + H(\{p_k\}) \quad (51)$$

which implies

$$|H(\sum_k p_k \varrho_k, \mathcal{P}) - \sum_k p_k H(\varrho_k, \mathcal{P})| \leq H(\{p_k\}) \quad (52)$$

Proof of theorem 1.

From lemma 2 we have that

$$\mathcal{M}(\varrho, \mathcal{P}) - \mathcal{M}((1 - \varepsilon)\varrho + \varepsilon\sigma, \mathcal{P}) \leq 4\varepsilon \log_2 d + H(\varepsilon) \quad (53)$$

then

$$\mathcal{M}(\varrho, \mathcal{P}) \leq \mathcal{M}((1 - \varepsilon)\varrho + \varepsilon\sigma, \mathcal{P}) + 4\varepsilon \log_2 d + H(\varepsilon) \quad (54)$$

Let \mathcal{P}_ϱ be a measurement achieving $\mathcal{M}(\varrho)$. Then

$$\mathcal{M}(\varrho, \mathcal{P}_\varrho) = \mathcal{M}(\varrho) \leq \mathcal{M}((1 - \varepsilon)\varrho + \varepsilon\sigma, \mathcal{P}) + 4\varepsilon \log_2 d + H(\varepsilon) \leq \mathcal{M}((1 - \varepsilon)\varrho + \varepsilon\sigma) + 4\varepsilon \log_2 d + H(\varepsilon) \quad (55)$$

so

$$\mathcal{M}(\varrho) - \mathcal{M}((1 - \varepsilon)\varrho + \varepsilon\sigma) \leq 4\varepsilon \log_2 d + H(\varepsilon) \quad (56)$$

Analogously we can show that

$$\mathcal{M}(\varrho) - \mathcal{M}((1 - \varepsilon)\varrho + \varepsilon\sigma) \geq -(4\varepsilon \log_2 d + H(\varepsilon)) \quad (57)$$

Inequalities (56) and (57) together give us inequality (47), which is equivalent to the one from theorem 1, which ends the proof.

CONCLUSION

We have introduced a new quantity - an entropic entanglement parameter (\mathcal{E} -parameter), which has the same feature as coherent information: both can be negative. More precisely, we have shown, that for all separable states it is always nonpositive and indeed happens to be negative, which we have shown for maximally mixed state. Moreover, we have proved, that \mathcal{E} -parameter is asymptotically continuous and we have obtained upper and lower bounds for some classes of states. The \mathcal{E} -parameter is rather difficult to deal with, which is caused by its definition being a kind of so called "minmax". Note that, the parameter is not LOCC monotone. It follows from two facts: one that we can pass from any separable state to other one using LOCC operations and second that \mathcal{E} -parameter has not the same value for all separable states.

There are still many open questions. We would like to know how the value of \mathcal{M} can change if in definition we take supremum over POVMs instead of only von Neumann measurements, in particular, whether is it possible to obtain infinity. There is also an interesting question whether the \mathcal{E} -parameter is, in general, bounded from below by coherent information, which we have proven for some classes of states.

Finally, we believe that entropic entanglement parameter may reveal some new feature of entanglement as it feels the structure of the state and is connected with complementarity between eigenbasis of an entangled state and a product one.

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